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Higher-dimensional quantum Hall effect in string theory

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Abstract: We construct a string theory realization of the $4+1$d quantum Hall effect recently described by Zhang and Hu. The string theory picture contains coincident D4-branes forming an S^4 and having D0-branes (i.e. instantons) in their world-volume. The charged particles are modelled as string ends. Their configuration space approaches in the large-N limit a $\mathbf{CP^3}$, which is an S^2 fibration over S^4, the extra S^2 being made out of the Chan-Paton degrees of freedom. An alternative matrix theory description involves the fuzzy S^4. We also find that there is a hierarchy of quantum Hall effects in odd-dimensional spacetimes, generalizing the known cases in $2+1$d and $4+1$d.

Keywords: D-branes, M(atrix) Theories, Brane Dynamics in Gauge Theories, Non-Commutative Geometry.

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Contents

1. Introduction

Recently, condensed matter physicists Zhang and Hu found a remarkable theoretical construction of an interesting physical phenomenon — a $4+1$ dimensional quantum Hall effect [1] on an S^4. In this note we will describe a simple brane construction which reproduces the same physics.

In fact, there are many connections between string theory and the ordinary $2+1$d quantum Hall physics which were found during the last two years. The first of them was a construction of Bernevig, Brodie, Susskind and Toumbas [2] reproducing the quantum Hall effect on a sphere (see figure 1), followed by a series of papers including [3]. Another line of progress (including [4]) was started by the proposal of Susskind [5] that the granular structure of the quantum Hall fluid can be captured by making the ordinary Chern-Simons description non-commutative. This model can also be obtained from the lagrangian of a charged particle moving in magnetic field by replacing its coordinates by matrices [5], in the spirit of matrix theory for D0-branes.

The $4+1$d quantum Hall system of [1] is the dynamics of particles in a large representation of SU(2) moving under the influence of the homogeneous instanton of SU(2) on an S^4. An important point in making a connection of this system to string theory is to translate the SU(2) dynamics of [1] into the dynamics of fundamentals of U(n) moving in a background U(n) field.

The string theory construction itself is a close analog of [2] (see figure 2). The four-sphere is modelled by a stack of coincident spherical D4-branes with a homogeneous instanton of $U(n)$ in their world-volume. This instanton has instanton number $N = \frac{1}{6}(n-1)n(n+1)$, and can be also thought of as N D0-branes in the D4-brane world-volume. Note that precisely this system was constructed also in matrix theory [6], and is referred to as the 'fuzzy four-sphere'. The charged particles themselves can be modelled as the ends of fundamental strings connecting the spherical D4-branes to a stack of flat D4-branes placed at the center of the S^4. This system by itself will not be stable, but we will briefly comment on possible ways of stabilizing it. Note also that the question of the string end statistics is subtle. However, we can expect logic similar to [2] to apply here as well.

The string theory point of view makes the analogy between the ordinary $2+1$d and the $4+1$d quantum Hall systems rather close — one can be obtained from another simply by changing the dimensionality of the D-branes. The QHE corresponds to the dynamics of strings ending on either a fuzzy S^2 or a fuzzy S^4.

This point of view also provides an insight into the question of generalizing the QHE to dimensions higher than four. This can be achieved simply by replacing the fuzzy S^2 or S^4 by a fuzzy S^6 or S^8. It is probably very hard to stabilize such systems in string theory. On the other hand, we can also think of them as a simple quantum mechanics in a non-dynamical background field, out of the framework of string theory, in which case such problems do not arise. Then we can even talk about QHE on an S^{2k} for $k > 4$. The corresponding homogeneous background gauge field can be found for example using the techniques of [7, 8].

2. Review: $2+1$d quantum Hall effect on an S^2

Here we briefly mention some basic properties of charged particles moving in a constant magnetic field on S^2 [9]. The magnetic field is sourced by a magnetic monopole inside the S^2, such that the total magnetic flux through the sphere is $n-1 \equiv 2I \in \mathbf{Z}$, say $I > 0$. The single-particle hamiltonian can be written as

$$H = \frac{1}{2MR^2} \sum_{\mu<\nu} \Lambda_{\mu\nu}^2 \,, \tag{2.1}$$

where $\mu, \nu = 1, 2, 3$; $\Lambda_{\mu\nu} \equiv -i(x_\mu D_\nu - x_\nu D_\mu)$; and $\tilde{x}_\mu \equiv R x_\mu$ are the embedding coordinates of the S^2. This hamiltonian is invariant under SO(3) transformations generated by $L_{\mu\nu} \equiv \Lambda_{\mu\nu} + I\epsilon_{\mu\nu\rho}x_\rho$. The spectrum follows from the relation $\sum_{\mu<\nu} \Lambda_{\mu\nu}^2 = \sum_{\mu<\nu} L_{\mu\nu}^2 - I(I+1)$ and from the fact that $\sum_{\mu<\nu} L_{\mu\nu}^2$ has eigenvalues $l(l+1)$ with $l = I,\ I+1,\ I+2, \ldots$

The lowest Landau level states transform in the $l = I$ representation of SO(3), so their degeneracy is $n = 2I + 1$. Lowest Landau states localized about a particular point x_μ on the S^2 are eigenstates of $x_\mu I_\mu$, where I_μ are the $l = I$ representation matrices of SO(3). I_μ can be also viewed as the uplift of 3d euclidean Dirac gamma matrices (i.e. the Pauli matrices) from the $l = 1/2$ representation to the $2I$-th symmetric tensor power of the $l = 1/2$ representation (i.e. to the $l = 2I$ representation). In other words

$$I_\mu \sim (\sigma_\mu \otimes 1 \otimes 1 \cdots \otimes 1 + 1 \otimes \sigma_\mu \otimes 1 \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes 1 \cdots \otimes \sigma_\mu)_{\mathrm{sym}} . \tag{2.2}$$

This point of view will be useful in section 5.3.

2.1 The string theory picture

There is a cool string theory realization of this kind of quantum Hall effect due to Bernevig, Brodie, Susskind and Toumbas [2]. To construct their quantum Hall system (see figure 1), one starts with a spherical D2-brane and dissolves $n-1$ D0-branes in it. Then one takes K coincident D6-branes extended in the directions perpendicular to the ones where the D2 lives, and moves them to the center of the two-sphere. When the D6-branes cross the D2's, the Hanany-Witten effect [10] produces fundamental strings stretching between them. This configuration can be in equilibrium due to repulsion of D6-branes and D0's. We know that D0-branes behave as magnetic flux in the world-volume of D2's and that string ends are charged under the world-volume gauge field. As a result, the low-energy physics is essentially the one described in the previous paragraph.

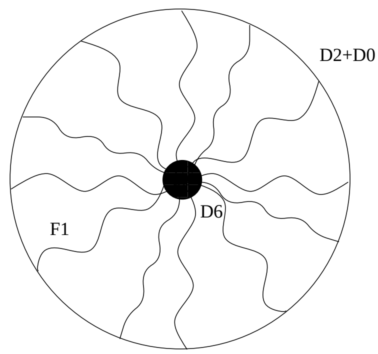

Figure 1: This picture, stolen from [2], shows a string theory realization of the $2+1$d quantum Hall effect on a two-sphere.

The D2 with D0 brane charge can be also thought of as $n-1$ D0-branes expanded into a spherical configuration which has an induced local D2-brane charge [11], provided the separation between neighboring D0's is much smaller than the string length, and $n \gg 1$. In this picture, we can think of the strings as being connected to the constituent D0-branes. Assume for simplicity that $K = 1$. The lowest Landau levels should correspond to unexcited stings, so their degeneracy should be equal to $n-1$. This indeed agrees with the original picture in the previous section, up to a subleading correction — a unit shift of n.

3. Review: $4+1$d quantum Hall effect on an S^4

The 4+1d quantum Hall physics of [1] is the quantum mechanics of massive charged particles moving on a four-sphere in a time-independent SU(2) gauge field. This field is taken to be the homogeneous SU(2) instanton (i.e. a gauge configuration of a non-vanishing second Chern class, with instanton number one).

The charged particles are in some definite representation of the gauge group corresponding to some value $I \equiv (n-1)/2$ of the SU(2) 'isospin', and can be described by n-component vectors. It was found in [1] that eventually, one has to take n very large to obtain a reasonable thermodynamic limit.

3.1 The second Hopf map

To parameterize the four-sphere, we will use (slightly modified) conventions of [1]. The embedding coordinates $\tilde{x}_\mu = Rx_\mu$ ($\mu = 1,\ldots,5$, $x_\mu x_\mu = 1$) of the S^4 in a 5d euclidean space can be written as

$$x_\mu = \bar{\Psi}_\alpha (\Gamma_\mu)_{\alpha\alpha'} \Psi_{\alpha'}\,, \qquad \bar{\Psi}_\alpha \Psi_\alpha = 1\,. \tag{3.1}$$

Here, the Γ_μ are 4×4 euclidean Dirac gamma matrices, $\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$, and Ψ_α is a four component complex spinor, the fundamental spinor of SO(5). Obviously, there is a redundancy in parameterizing the S^4 with Ψ_α. Indeed, if we choose the gamma matrices as

$$\Gamma^i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \qquad i = 1,\ldots,3, \qquad \Gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.2}$$

we can write Ψ_α in the following form,

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{\frac{1+x_5}{2}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \sqrt{\frac{1}{2(1+x_5)}} (x_4 - i x_i \sigma_i) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{3.3}$$

with some two-component complex spinor (u_1, u_2) satisfying $\bar{u}_\sigma u_\sigma = 1$. Varying u_σ changes Ψ_α, but leaves x_μ invariant. The space of different u_σ is an S^3, Ψ_α span an S^7, and as we said, x_μ are coordinates on an S^4. This construction of S^7 as a fibration of S^3 over S^4 is known as the second Hopf map. If we mod out this S^7 by the U(1) phase rotations of Ψ_α, we obtain $\mathbf{CP^3}$ as an S^2 fibration over S^4.

3.2 The quantum Hall mechanics

The gauge field of the homogeneous SU(2) instanton on the S^4, i.e. of the Yang monopole [12], can be written as

$$A_\mu = \frac{a_\mu}{R}\,; \qquad a_A = -\frac{i}{1+x_5} \eta_{iAB} x_B \frac{\sigma_i}{2}\,, \qquad a_5 = 0\,, \tag{3.4}$$

$$\eta_{iAB} = \epsilon_{iAB4} + \delta_{iA}\delta_{4B} - \delta_{iB}\delta_{4A}\,, \qquad A, B = 1,\ldots,4 \tag{3.5}$$

where σ_i are the Pauli matrices. (The gauge connection is $a_\mu dx_\mu$, with the restriction $x_\mu dx_\mu$ =0). This potential can also be obtained as $\bar{u}_\sigma a_{\sigma\sigma'} u_{\sigma'} = \bar{\Psi}_\alpha d\Psi_\alpha$, which means that (3.4) is precisely what we get if we think of the S^3 in the second Hopf map as an SU(2) gauge field living on the S^4.

The single-particle hamiltonian of [1] is simply the kinetic energy of the charged particles plus the interaction with the SU(2) gauge field,

$$H = \frac{1}{2MR^2} \sum_{\mu<\nu} \Lambda_{\mu\nu}^2\,. \tag{3.6}$$

Here, M is the mass of the particle, and $\Lambda_{\mu\nu} = -i(x_\mu D_\nu - x_\nu D_\mu)$, with some appropriate covariant derivatives D_μ whose form depends on the SU(2) representation chosen. In general, H will be an $n\times n$ matrix, with $I \equiv (n-1)/2$ being the value of the SU(2) 'isospin'.

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The hamiltonian (3.6) is SO(5) invariant, though the usual SO(5) action has to be accompanied by an extra isospin rotation. In other words, the SO(5) generators are

$$L_{\mu\nu} \equiv \Lambda_{\mu\nu} - i f_{\mu\nu}, \quad f_{\mu\nu} \equiv [D_\mu, D_\nu] \,. \tag{3.7}$$

As a result, the energy eigenstates form SO(5) representations. Any SO(5) representation can be labelled by two integers $r_1 \geq r_2 \geq 0$, the row lengths of the corresponding Young diagram.[1] The dimension of the representation is

$$D(r_1, r_2) = \frac{1}{6}(r_1 + r_2 + 2)(r_1 - r_2 + 1)(3 + 2r_1)(1 + 2r_2) \,. \tag{3.8}$$

Expressed in terms of the variables of [1],

$$p = r_1 + r_2 \,, \qquad q = r_1 - r_2 \,, \qquad p \geq q \geq 0 \,, \tag{3.9}$$

this becomes

$$\tilde{D}(p, q) = (1 + q)(1 + p - q)\left(1 + \frac{p}{2}\right)\left(1 + \frac{p + q}{3}\right) \,. \tag{3.10}$$

The variables p and q satisfy [1]

$$p - q = 2I \,, \qquad I \equiv \frac{n - 1}{2} \,, \tag{3.11}$$

implying

$$r_1 \geq r_2 = I \,. \tag{3.12}$$

The reason why there is a restriction on the value of r_2 is that the particles are point-like, and therefore they cannot have two independent 'angular momenta', i.e. two independent charges under the Cartan subalgebra of the SO(5). The value of r_2 is non-zero just because the generators of this SO(5) are not exactly equal to the dynamical angular momenta if $I \neq 0$.

Using the identity

$$H = \frac{1}{2MR^2} \sum_{\mu<\nu} \Lambda_{\mu\nu}^2 = \frac{1}{2MR^2} \left(\sum_{\mu<\nu} L_{\mu\nu}^2 - 2I(I + 1) \right) \,, \tag{3.13}$$

one can easily find the energy corresponding to a given SO(5) representation,

$$E(I, q) = \frac{1}{2MR^2} \left(2I + (2I + 3)q + q^2 \right) \,. \tag{3.14}$$

We see that q plays the role of the Landau level index. The degeneracy of the lowest Landau level ($q = 0$) is

$$\tilde{D}(n - 1, 0) = \frac{1}{6} n(n + 1)(n + 2) \,. \tag{3.15}$$

[1] A nice and succinct review of SO($2k + 1$) and SO($2k$) representations can be found in [7].

4. U(n) interpretation of the $4+1$d quantum Hall effect

To make any connection to string theory, we have to get rid of the extremely high gauge representations of the charged particles. This can be done very easily. Note that the gauge connection of the charged particles due to (3.4) can be obtained from (3.4) by replacing $\frac{1}{2}\sigma_i$ with appropriate generators I_i of the SU(2) Lie algebra [1], $[I_i, I_j] = i\epsilon_{ijk}I_k$,

$$A_\mu = \frac{a_\mu}{R}\,; \qquad a_A = -\frac{i}{1+x_5}\eta_{iAB}x_B I_i\,, \qquad a_5 = 0\,. \tag{4.1}$$

The particles are described by n-component vectors, and both (3.6) and (4.1) are $n \times n$ matrices. Thus, the system is equivalent to particles in the fundamental representation of U(n) moving under the influence of a U(n) gauge field, given by (4.1). The equivalence is possible only because here we are treating all the gauge fields as a fixed background. Of course, if they were dynamical, there would be a big difference between SU(2) and U(n) gauge fields.

The gauge configuration (4.1) is actually the homogeneous instanton solution of U(n) with instanton number

$$N = \frac{1}{6}(n-1)n(n+1)\,. \tag{4.2}$$

A review of these homogeneous instantons can be found for example in [13, appendix B]. From string theory we know that for large R we can think of this gauge field as N D0-branes in the world-volume of n coincident spherical D4-branes.

5. String theory construction of the $4+1$d quantum Hall effect

Having identified the brane construction of the gauge field on the four-sphere in the previous section, it is easy to model the full system, including the 'electrons'. This can be done in a way analogous to [2]. We start with n coincident spherical D4-branes and spread $\frac{1}{6}(n-1)n(n+1)$ D0-branes in their world-volume. Say these D4-branes are extended in $\tilde{x}_\mu$, $\mu = 1, \ldots, 5$. Consider also m flat infinite D4-branes far from the four-sphere, extended in $\tilde{x}_\mu$, $\mu = 6, \ldots, 9$. Now move the m D4-branes to the center of the four-sphere. The Hanany-Witten effect [10] produces mn fundamental strings connecting the D4-branes at the center to the ones forming the four-sphere. The string ends are fundamentals of the D4-brane U(n) gauge group, which is precisely what we need to interpret them as the 'electrons' of [1].

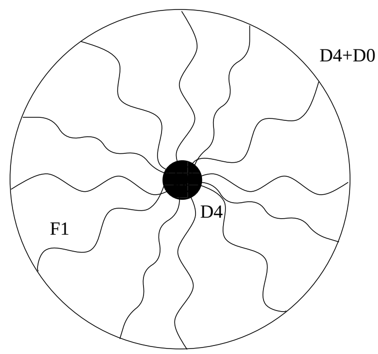

Figure 2: The string theory picture of the $4+1$d quantum Hall effect on an S^4 can be obtained from the one at 1 simply by changing the dimensionality of the D-branes.

Since the system breaks supersymmetry, it cannot be fully stable. It is not even metastable, and it will immediately start to collapse. Of course, if we want to study the stringy quantum Hall physics of this system, we have to make it at least metastable or very slowly decaying. One of the possible ways might be placing the whole system into a fluxbrane [14] with a non-zero

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Ramond-Ramond field strength $F_6 = dC_5$ with Lorentz SO(5,1) and rotational SO(4) symmetry [15]. (A similar set-up for two-spheres was studied in [16].) If we make the RR field strong enough, the spherical D4-branes want to expand. On the other hand, the field strength of the RR field goes to zero at infinity, so one might expect that the system will reach some equilibrium radius. Of course, the stability with respect to non-spherical deformations will be important, too. In this paper, we will leave this question open.

5.1 Fuzzy four-sphere interpretation

There is a nice interpretation of the spherical D4-branes carrying D0-brane charge — the non-commutative (or fuzzy) four-sphere [17]. Such a system can be constructed in matrix theory, where one starts with $\frac{1}{6}(n-1)n(n+1)$ D0-branes and expands them into an S^4 [6]. The local D4-brane charge of such a configuration was found to be n.

As a result, the system can be described either in terms of D4-branes with a D0-brane charge, or in terms of D0-branes expanded into a fuzzy S^4. Just like in [11], these two pictures should agree at the leading order in n, but there is no reason to expect that also the subleading corrections are the same.

Now, we would like to see what implications the matrix fuzzy S^4 picture has for the quantum Hall physics. As in [2], one can think of the fundamental string as being connected to the D0-branes forming the S^4. Let us for simplicity assume that there is just one D4 at the center of the S^4 (i.e. $m = 1$). We can expect that there is one unexcited string state for each D0-brane. Because unexcited strings have the lowest possible energy, they should correspond to the lowest Landau levels. As a result, we expect the lowest Landau level to be $\frac{1}{6}(n-1)n(n+1)$-fold degenerate. Indeed, at the leading order in n, this agrees with (3.15).

5.2 The magic geometry of the fuzzy S^4

We have seen that the 4 + 1d QHE is in a good sense really 4 + 1-dimensional — it is the physics of U(n) fundamentals on an S^4. However, if we kept n large and asked the fundamentals what configuration space they live in, they would describe it (especially after reading section 3.1) as an S^2 fibration over S^4, the S^2 fibre representing the possible orientations of the SU(2) 'isospin'. They could also mention that the space is actually a $\mathbf{CP^3}$, if they wanted to be as precise as Edward Witten, who pointed out this interesting fact. The $\mathbf{CP^3}$ can be viewed[2] as $\mathrm{Sp}(2)/\mathrm{SU}(2)\times\mathrm{U}(1)$, where $\mathrm{SU}(2)\times\mathrm{U}(1)\subset\mathrm{SU}(2)\times\mathrm{SU}(2)=\mathrm{Sp}(1)\times\mathrm{Sp}(1)\subset\mathrm{Sp}(2)$. Alternatively, it is also the bundle of unit anti-self-dual two-forms on S^4.

This agrees with the matrix theory calculations of [8], where it was found that the full matrix algebra of the fuzzy S^4 approaches in the large-N limit the algebra of functions on $\mathrm{SO}(5)/\mathrm{U}(2)=\mathrm{Sp}(2)/\mathrm{SU}(2)\times\mathrm{U}(1)$.

Associated with these two points of view are two alternative descriptions of the same physics — a 6 + 1-dimensional [18] and a 4 + 1-dimensional. This kind of duality was discussed in [8] in the context of ordinary field theory on the fuzzy S^4. Probably the most interesting aspect of this correspondence is that from the D4-brane point of view, the two extra dimensions are made out of the D4-brane gauge fields.

[2] I would like to thank Luboš Motl for discussions about this point.

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5.3 How to see the fuzzy S^4 without using string theory

The fuzzy four-sphere structure of the lowest Landau level ($q = 0$, $p = 2I$) of [1] can be seen quite easily even without the machinery of string theory. The single particle states $\tilde{\Psi}$ transform in the ($r_1 = I$, $r_2 = I$) of SO(5), which is the $2I$-th symmetric tensor power of the fundamental spinor Ψ of SO(5), i.e. $(1/2, 1/2)$. $\tilde{\Psi}$ can be thought of as a vector with $P \equiv \frac{1}{6}(p+1)(p+2)(p+3)$ components, because the lowest Landau level is P-fold degenerate.

In analogy to section 2, the lowest Landau states localized about a certain point x_μ on the S^4 are eigenstates of $x_\mu \tilde{X}_\mu$. The operators X_μ can be obtained by uplifting the gamma matrices Γ_μ from the $(1/2, 1/2)$ representation to $(I, I) = (p/2, p/2)$, cf. also (3.1). In other words,

$$X_\mu \sim (\Gamma_\mu \otimes 1 \otimes 1 \cdots \otimes 1 + 1 \otimes \Gamma_\mu \otimes 1 \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes 1 \cdots \otimes \Gamma_\mu)_{\mathrm{sym}} \,. \tag{5.1}$$

Moreover, since the wave-functions $\tilde{\Psi}$ transform in the (I, I) of SO(5), we have

$$L_{\mu\nu} \tilde{\Psi} = -\frac{i}{4}[X_\mu, X_\nu] \tilde{\Psi} \,, \tag{5.2}$$

and

$$H\tilde{\Psi} = \frac{1}{2MR^2}\left(\sum_{\mu<\nu} L_{\mu\nu}^2 - I(I+1)\right)\tilde{\Psi} = \frac{1}{2MR^2}\left(-\sum_{\mu<\nu}\frac{1}{16}[X_\mu, X_\nu]^2 - I(I+1)\right)\tilde{\Psi} \,. \tag{5.3}$$

Now, the connection to the fuzzy four-sphere became obvious. In (5.1) we recognize the matrices representing the fuzzy S^4 [6], and (5.2) are the generators of its rotations. However, it is hard to judge from the present calculations whether there is any deeper reason why (5.3) resembles the usual matrix theory hamiltonian.

6. Generalization to higher dimensions

We have seen that the QHE on 2d or 4d spherical surfaces corresponds to the quantum mechanics of charged particles moving in the gauge field of the fuzzy S^2 or S^4. Higher-dimensional fuzzy spheres have been constructed as well, at least in matrix theory. Physics of charged particles (or string ends) moving in their gauge field naturally generalizes the 2 + 1d and 4 + 1d QHE. Let us briefly mention what properties of these quantum Hall systems can be inferred from the matrix theory results of [7, 8].

The fuzzy S^6 consists of

$$N_6 = \frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5) \tag{6.1}$$

D0-branes and its D6-brane charge is

$$D_6 = \frac{1}{6}(n+1)(n+2)(n+3) \,. \tag{6.2}$$

Therefore the 6+1d QHE should be the dynamics of the fundamentals of $\mathrm{U}(D_6)$ in a homogeneous background $\mathrm{U}(D_6)$ gauge field with a non-zero third Chern class. The degeneracy of the lowest Landau level should be N_6, up to subleading shifts of n. The case of the fuzzy S^8 is analogous, with

$$N_8 = \frac{1}{302400}(n+1)^2(n+2)^2(n+3)^4(n+4)^4(n+5)^4(n+6)^2(n+7)^2\,, \tag{6.3}$$

and

$$D_8 = \frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5)\,. \tag{6.4}$$

In general for a fuzzy S^{2k}, the configuration space of the fundamentals should closely approximate $\mathrm{SO}(2k+1)/\,\mathrm{U}(k)$ at large n.

7. Conclusions

We have described a string theory system reproducing the quantum Hall effect of Zhang and Hu. Moreover, we have seen that there is an extremely close relationship between this $4+1$d and the ordinary spherical $2+1$d quantum Hall system — they are the first two elements in a hierarchy of quantum Hall systems on even-dimensional spheres.

Of course, many questions related to the $4+1$d quantum Hall effect are still open — for example the issue finding an appropriate non-commutative Chern-Simons description, or understanding well the physics of the edge excitations. It is natural to expect that string theory will play an important role in answering these questions, judging from the many recently discovered connections between string theory and the ordinary quantum Hall effect.

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